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Approximating the Probability Distribution of Total System Cost

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From Garvey, P. R., *"Probability Methods for Cost Uncertainty Analysis-A Systems Engineering Perspective"*, published by Marcel Dekker, Inc., 270 Madison Avenue, New York, NY, 10016-0602, 1999.

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Main Points

- The probability distribution of a system's total cost can *often* be approximated analytically by a "simple formula"
- Analytical approximations provide insights into cost risk drivers and other influences not readily seen in empirical distributions (e.g., Monte Carlo simulations) of a system's total cost
- Five circumstances are presented for identifying when the probability distribution of a system's total cost can be approximated analytically
- The normal or the lognormal distribution will almost always provide a good analytical approximation to the underlying distribution function of a system's total cost

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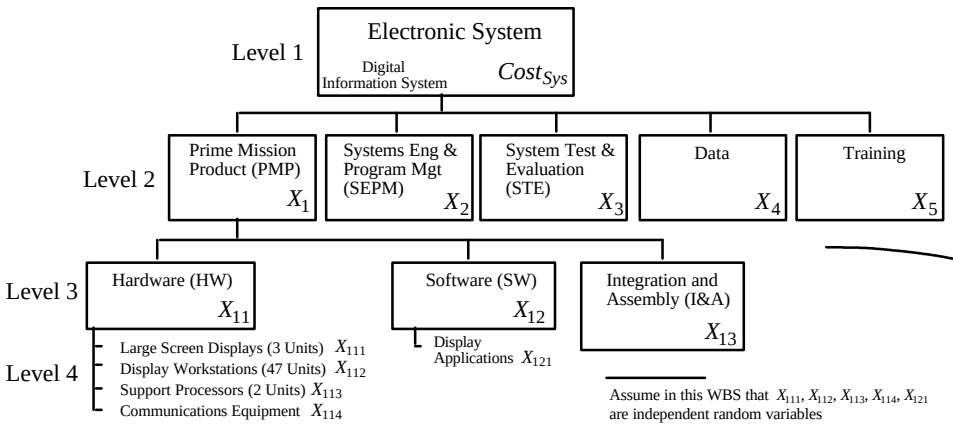
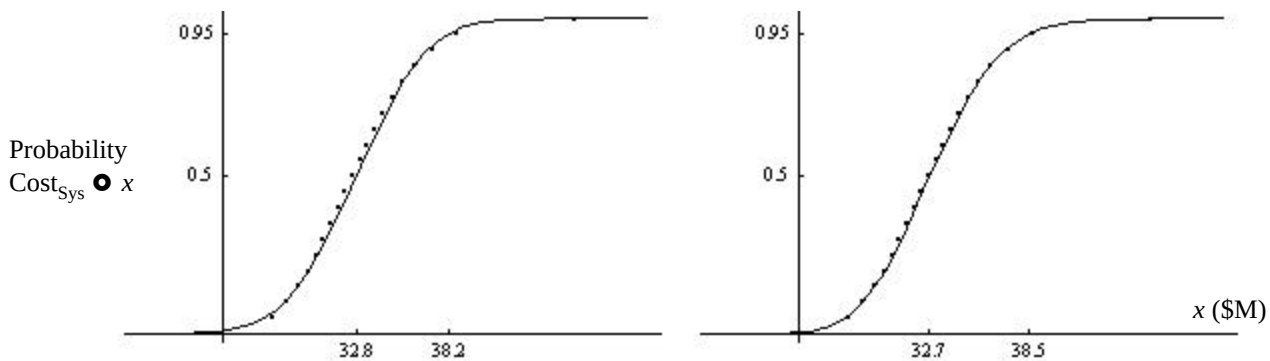
A Work Breakdown Structure Perspective



$$Cost_{Sys} = Cost_{PMP} + \sum_{i=2}^{n} X_i \quad (6-88)$$

where $Cost_{PMP} = X_1$



Solid Line- Assumed Normal
Dots- Monte Carlo Simulation

Solid Line- Assumed LogNormal
Dots- Monte Carlo Simulation

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Circumstance A

Circumstance A

If (in equation 6-88) the distribution function of $Cost_{PMP}$ *is normal and* $X_2, X_3, X_4, \ldots, X_n$ *are linear functions of* $Cost_{PMP}$, *such as* $X_i = a_i\, Cost_{PMP}$ *where* $a_i \geq 0$ $(i = 2, \ldots, n)$, *then the distribution function of* $Cost_{Sys}$ *is normal with mean*

$$(1 + a_2 + a_3 + \ldots + a_n) E\left(Cost_{PMP}\right)$$

and variance

$$(1 + a_2 + a_3 + \ldots + a_n)^2 Var\left(Cost_{PMP}\right)$$

Circumstance A is a direct consequence of the following:

Proposition 6-1 If X is a normal random variable and $Y = a\,X$, where a is a constant, then the distribution function for Y is normal with mean $a\,E(X)$ and variance $a^2 Var(X)$.

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Circumstance B

Circumstance B

If (in equation 6-88) $Cost_{PMP}$ *and* $X_2, X_3, X_4, \ldots, X_n$ *are independent random variables and each are normally distributed, then the distribution function of* $Cost_{Sys}$ *is normal with*

$$mean \quad E(Cost_{PMP}) + \sum_{i=2}^{n} E(X_i) \text{ and variance } Var(Cost_{PMP}) + \sum_{i=2}^{n} Var(X_i)$$

Circumstance B is a direct consequence of the following:

Proposition 6-2 If $X_1, X_2, X_3, \ldots, X_k$ are independent *normally* distributed random variables and $Y = X_1 + X_2 + X_3 + \ldots + X_k$, then, regardless of the size of *k*, the distribution function of *Y* is normal with mean $\sum_{i=1}^{k} E(X_i)$ and variance $\sum_{i=1}^{k} Var(X_i)$.

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Circumstance C

Circumstance C

Suppose (in equation 6-88) $Cost_{PMP}$ *is normal and* $Cost_{PMP}, X_2, X_3, X_4, \ldots, X_n$ *are independent random variables. Furthermore, suppose* $X_2, X_3, X_4, \ldots, X_n$ *are not necessarily each normally distributed. If the number of cost element costs in the sequence* $X_2, X_3, X_4, \ldots, X_n$ *is sufficiently large with no* X_i $(i = 2, \ldots, n)$ *dominating in standard deviation, then the distribution function of* $Cost_{Sys}$ *is approximately normal with*

$$mean \quad E(Cost_{PMP}) + \sum_{i=2}^{n} E(X_i) \text{ and variance } Var(Cost_{PMP}) + \sum_{i=2}^{n} Var(X_i)$$

Circumstance C is a combination of the central limit theorem and proposition 6-2. To see this, recall from equation 6-88 $Cost_{Sys}$ can be written as

$$Cost_{Sys} = Cost_{PMP} + \sum_{i=2}^{n} X_i$$

If the distribution function for $Cost_{PMP}$ is normal and the distribution function of the sum $\sum_{i=2}^{n} X_i$ is approximately normal (by the central limit theorem), then $Cost_{Sys}$ is the sum of two normally distributed random variables. In circumstance C, $Cost_{PMP}$ and $\sum_{i=2}^{n} X_i$ are independent. Thus, from proposition 6-2, the distribution function of $Cost_{Sys}$ is normal.

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Circumstance D

Circumstance D

Suppose (in equation 6-88) $Cost_{PMP}$ *is normal. Suppose the sequence* $X_2, X_3, X_4, \ldots, X_n$ *contains some cost element costs correlated to* $Cost_{PMP}$ *(with correlation coefficient* r_{Cost_{PMP}, X_i}*) and some that are uncorrelated to* $Cost_{PMP}$*. Furthermore, suppose those that are uncorrelated to* $Cost_{PMP}$ *are mutually independent random variables. If the number of* X_i*'s* $(i \geq 2)$ *uncorrelated to* $Cost_{PMP}$ *is sufficiently large, with none dominating in standard deviation, then the distribution function of* $Cost_{Sys}$ *is approximately normal with*

$$\text{mean } E(Cost_{PMP}) + \sum_{i=2}^{n} E(X_i)$$

$$\text{and variance } Var(Cost_{PMP}) + \sum_{i=2}^{n} Var(X_i) + 2\sum_{i=2}^{n} r_{Cost_{PMP}, X_i} s_{Cost_{PMP}} s_{X_i}$$

If the X_i*'s* $(i \geq 2)$ *uncorrelated to* $Cost_{PMP}$*, which are also mutually independent, are each normally distributed, then the distribution function of* $Cost_{Sys}$ *is approximately normal with mean and variance given above; here it is not required there be sufficiently many of these* X_i*'s.*

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Circumstance E

Circumstance E

Suppose (in equation 6-88) $Cost_{PMP}, X_2, X_3, X_4, \ldots, X_n$ *are independent random variables. Furthermore, suppose* $Cost_{PMP}, X_2, X_3, X_4, \ldots, X_n$ *are* <u>*not*</u> *necessarily each normally distributed. If the number of cost element costs in the sequence* $Cost_{PMP}, X_2, X_3, X_4, \ldots, X_n$ *is sufficiently large with none dominating in standard deviation, then (by the central limit theorem) the distribution function of* $Cost_{Sys}$ *is approximately normal with*

$$\textit{mean } E(Cost_{PMP}) + \sum_{i=2}^{n} E(X_i) \textit{ and variance } Var(Cost_{PMP}) + \sum_{i=2}^{n} Var(X_i)$$

In all but the last circumstance discussed above, the distribution function for $Cost_{PMP}$ was given to be normal. This is common in electronic systems. The normality of $Cost_{PMP}$ is primarily driven by the central limit theorem, where $Cost_{PMP}$ typically reflects the sum of many *independent* hardware and software costs.

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In Summary

- Cost analysts should understand the mathematical relationships they define in a system's WBS to see whether analytical approximations to the probability distribution function of $Cost_{Sys}$ can be argued
- A class of circumstances exist that indicate when a *normal* distribution approximates the probability distribution function of $Cost_{Sys}$; recognizing when these circumstance apply, can save time and resources needed for complex Monte Carlo simulations
- It is typical to have a mixture of independent and correlated cost element costs in a system's WBS; howver, because of the central limit theorem, the greater the number of independent cost element costs the more it is that the distribution function of $Cost_{Sys}$ is approximately normal

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In Summary

- The central limit theorem is very powerful; it does not take many independent cost element costs for the distribution function of $Cost_{Sys}$ to move towards normality - even when correlation is present between some cost element costs
- Such a move is evidenced when
  - A sufficient number of independent cost element costs are summed
  - When no cost element's cost distribution has a much larger standard deviation than the standard deviations of the other cost element cost distributions
- When conditions in the WBS result in $Cost_{Sys}$ being positively skewed, then the *lognormal* often approximates the probability distribution function of $Cost_{Sys}$

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